J. Appl. Maths Mechs, Vol. 55, No. 4, pp. 511-516, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00

ORIENTED MANIFOLDS AND THE CONTROLLABILITY OF DYNAMICAL SYSTEMS*

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(Received 2 March 1990)

A method of oriented manifolds (OMs) /1/ is used to investigate non-linear autonomous control systems. Assuming that the boundary of the OM is differentiable, its equation is found and is used to obtain the necessary conditions for controllability and estimates of the controllable domain when there are restrictions on the control.

*1. **A theorem on the controllability of non-linear systems.*** We shall study dynamical systems described by the following ordinary differential equations:

$$x^{\cdot} = f(x, u) \tag{1.1}$$

where x is the phase vector and u is the control vector in a time interval $T = [0, \infty)$ and a domain $D = \{x\}$, which we assume to be a connected n-dimensional C^r-manifold $(r \geqslant 2)$. The permitted controls are bounded measurable functions of time $u = u(t)$ taking values in some set $U \subseteq R^m$. We shall also assume that $\forall u \in U$ the function $f(x, u)$ is $(r-1)$ times continuously differentiable on $D \times \bar{U}$.

We introduce the following definitions /2/.

Definition 1. We say that the point $x_1 \in D$ is reachable from the point $x_0 \in D$ if there exists a trajectory $x(t)$ of system (1.1) such that $x(0) = x_0$ and $x(t_1) = x_1$ for some $t_1 \in T$. The set of all points reachable from the point $x \in D$ will be called the positive orbit of the point x and denoted by $\mathrm{Or}^+ x$; the set $\mathrm{Or}^+ K = U\{\mathrm{Or}^+ x \colon x \in K\}$ is the positive orbit of the set $K \subset D$. The sets

$$\mathrm{Or}^- x = \{y \in D \colon x \in \mathrm{Or}^+ y\}, \quad \mathrm{Or}^- K = U\{\mathrm{Or}^- x \colon x \in K\}$$

are respectively called the negative orbits of the point x and the set K.

Definition 2. The set $K \subset D$ will be called oriented with respect to system (1.1) if $K = \mathrm{Or}^+ K$ or $K = \mathrm{Or}^- K$.

The simplest examples of oriented sets are D, $\varnothing$ and $\mathrm{Or}^+ K \ (K \subset D)$. In systems theory there are geometric objects of a similar nature: semi-permeable surfaces in the theory of differential games /3/, lines of single-sided hatchings /4/, locks and traps /5/ in the theory of non-linear systems etc.

Definition 3. Systems (1.1) will be called controllable if $\forall x \in D$, $\mathrm{Or}^+ x = D$.

This definition of controllability is identical with the definition of global controllability generally accepted in the literature.

Directly from these definitions we have the following theorem.

Theorem 1. Systems (1.1) is controllable if and only if there is no set $N \neq \varnothing, D$ that is oriented with respect to the system.

For an arbitrary control system, oriented sets can be made as complicated as desired, and hence Theorem 1 is not very useful. A more constructive approach to investigating controllability may be obtained using the concept of oriented manifolds (OMs) /1/. The following theorem holds /2/.

Theorem 2. System (1.1) is controllable if and only if there is no manifold $N \neq \varnothing, D$ that is oriented with respect to system (1.1).

*2. **Equations of oriented manifolds.*** The condition of orientability means that $\forall u \in U$ the velocity vectors $f(x, u)$ at the boundary points are directed into the exterior of the manifold if $K = \mathrm{Or}^- K$ or into the interior of the manifold if $K = \mathrm{Or}^+ K$. We construct tangent planes at points of the boundary, assuming it to be differentiable. We shall distinguish between the case of a manifold of complete dimension $(\dim N = n)$ and incomplete

*Prikl.Matem.Mekhan,55,4,639-646,1991

dimension ($\dim N = s < n$).

In the first case the boundary is governed by an equation $V(x) = 0$ ($V \in R^1$) and the tangent plane to the point by $x_0 - (x - x_0, \nabla V(x_0)) = 0$. The interior of the manifold is given by the direction of the vector $\nabla V(x_0)$ and it follows from the orientability that $(f(x_0, u), \nabla V(x_0)) \geqslant 0, \forall u \in U$ or $(f(x_0, u), \nabla V(x_0)) \leqslant 0, \forall u \in U$. These inequalities can be written as equalities, if we introduce a sign-constant function $G(x, u)$ and a continuous function $\lambda(x, u)$ in the domain $D \times U$ (omitting the index because of the arbitrariness of the point x_0):

$$(f(x, u), \nabla V(x)) = \lambda(x, u) V(x) + G(x, u), \quad \forall u \in U \tag{2.1}$$

We will formulate the result obtained in the form of the following theorem.

Theorem 3. Suppose a sign-constant function $G(x, u)$ and a continuous function $\lambda(x, u)$ exist in the domain $D \times U$ such that $\forall u \in U$ Eq.(2.1) has a solution $V(x)$ in the domain D. Then in the domain D there exists an OM, whose boundary is given by the equation $V(x) = 0$, and system (1.1) is not controllable in the domain D.

We consider the second case. The boundary of the OM is given by equations $V_i(x) = 0$, ($V_i \in R^1$) and the tangent plane to the point by $x_0 - (x - x_0, \nabla V_i(x_0)) = 0$ ($i = 1, \ldots, n - s$). The interior is determined by one of the vectors $\nabla V_i(x)$, say $\nabla V_1(x)$; here the equalities $V_2(x) = 0, \ldots, V_{n-s}(x) = 0$ should be satisfied. Then

$$(f(x_0, u), \nabla V_1(x_0)) \geqslant 0, \quad (f(x_0, u), \nabla V(x_0)) = 0 \quad (i = 2, \ldots, n - s)$$
$$\forall u \in U \text{ or } (f(x_0, u), \nabla V_1(x_0) \leqslant 0, \quad (f(x_0, u), \nabla V_i(x_0)) = 0$$

follows from the orientability condition. As in the first case, the last relation can be written in the form of a system of equalities, if a sing-constant function $G(x, u)$ and continuous functions $\lambda_{ij}(x, u)$ ($i, j = 1, 2, \ldots, n - s$) are introduced in the domain $D \times U$:

$$(f(x, u), \nabla V_i(x)) = \sum_{j=1}^{n-s} \lambda_{ij}(x, u) V_j(x) + G_i(x, u), \quad \forall u \in U \tag{2.2}$$
$$G_1(x, u) = G(x, u), \quad G_2 = \ldots = G_{n-s} = 0, \quad i = 1, 2, \ldots, n - s$$

This proves the following theorem.

Theorem 4. Suppose a sign-constant function $G(x, u)$ and continuous functions $\lambda_{ij}(x, u)$ ($i, j = 1, \ldots, n - s$) exist in the domain $D \times U$ such that $\forall u \in U$ the system of Eqs.(2.2) has solutions $V_1(x), \ldots, V_{n-s}(x)$ in the domain D. Then there exists an OM on the domain D, with boundary given by the equations $V_i(x) = 0$ ($i = 1, \ldots, n - s$) and system (1.1) is not controllable in the domain D.

Remarks. 1°. Eqs.(1.1), (2.2) and Theorems 3 and 4 generalize the corresponding equations and theorems for invariant manifolds of dynamical systems /6/ and control systems /1/ to the case of OMs of control systems.

2°. Equations similar to (2.1) occur in stability theory when the Lyapunov function method is employed. The existence of the Lyapunov function leads to the existence of a family of solutions $V(x) = c$ ($0 \leqslant c \leqslant c_0$) for the associated equation, which also ensures the stability or instability of the solution under study. A similar situation can also occur in a control problem; however this is not compulsory, because the existence of only one solution of (2.1) is sufficient for non-controllability, which could occur in a suitably prepared and possibly complex system.

3. *The controllability of non-linear systems without restriction on the control.* Theorems 3 and 4 reduce the problem of the controllability of system (1.1) to the study of the existence of solutions of systems of differential Eqs.(2.1) or (2.2). The latter problem is made more complex by the fact that the given equations contain the controlling parameter u, which in the case under consideration can take any values in R^m. This difficulty can be overcome, for example, with the help of a stratagem similar to the one introduced for the basis systems of /1/ when constructing invariant manifolds.

We will confine ourselves to Eqs.(2.1). At each point $x \in D$ we represent the vector $f(x, u)$ in the form of a linear combination of vector fields.

$$f(x, u) = \alpha_1(x, u) f_1(x) + \ldots + \alpha_l(x, u) f_l(x) + \alpha_{l+1}(x, u) f_{l+1}(x) + \ldots + \alpha_k(x, u) f_k(x) \tag{3.1}$$
$$\alpha_{l+1}(x, u) \geqslant 0, \ldots, \alpha_k(x, u) \geqslant 0, \quad \forall (x, u) \in D \times R^m$$

(the coefficients $\alpha_1(x, u), \ldots, \alpha_l(x, u)$ take both positive and negative values). Then if we have a solution $V(x)$ of the system of equations

$$(f_i(x), \nabla V(x)) = \lambda_i(x) V(x) + G_i \ (i = 1, \ldots, k) \tag{3.2}$$
$$G_1 = G_2 = \ldots = G_l = 0, \quad G_j = G_j(x) \quad (j = l+1, \ldots, k)$$

where $\lambda_i(x)$ $(i = 1, \ldots, k)$ are continuous functions in D and $G_j(x)$ are sign-constant functions of the same sign in D, then because of representation (3.1) the function $V(x)$ will be a solution of Eq.(2.1) in which the functions

$$\lambda(x, u) = \sum_{i=1}^{k} \alpha_i(x, u) \lambda_i(x), \quad G(x, u) = \sum_{j=l+1}^{k} \alpha_j(x, u) G_j(x)$$

satisfy the requirements of Theorem 3. This proves the following theorem.

Theorem 5. Suppose the function $f(x, u)$ in the domain $D \times R^m$ can be represented in the form (3.1) and functions $\lambda_i(x)$ $(i = 1, \ldots, k)$ continuous in D and sign-constant functions $G_j(x)$ $(j = l+1, \ldots, k)$ exist that are continuous and have the same sign in D such that system (3.2) has a solution in the domain D. Then there exists an OM for system (1.1) and system (1.1) is non-controllable in the domain D.

We remark that if the representation (3.1) is possible only for $l = n$, then an OM (and even more so, an invariant manifold) does not exist, which is obvious, because at each point motion is possible along all directions. If $l = k$, i.e. all $G_i = 0$ $(i = 1, \ldots, k)$, then the solution of system (3.2) determines an invariant manifold with boundary $V(x) = 0$. (The boundary itself is an invariant manifold). To obtain the conditions for its existence it can be effective to use /1/ the Jacobi bracket technique. Here the vector fields $f_i(x)$ and functions $\lambda_i(x)$ should be differentiable, for which it is necessary to require the differentiability of the function $f(x, u)$.

For system (3.2) of general form the existence of a solution of its first subsystem $(i = 1, \ldots, l)$ is verified by means of Jacobi brackets, as when finding invariant manifolds /1/. In the case of its co-existence the study of the full system (3.2) requires additional considerations, possibly not using Jacobi brackets.

Thus for two-dimensional systems oriented manifolds can exist if $f(x, u) = \alpha_1(x, u) f_1(x) + \alpha_2(x, u) f_2(x)$, $\alpha_2(x, u) \geqslant 0$. System (3.2) consists of two equations

$$(f_1(x), \nabla V) = \lambda_1(x) V, \quad (f_2(x), \nabla V) = \lambda_2(x) V + G(x) \tag{3.3}$$

Assuming the function $f(x, u)$ to be differentiable a sufficient number of times, the functions $f_1(x)$, $f_2(x)$ are also chosen to be differentiable and linearly independent, i.e. $\det(f_1(x), f_2(x)) \neq 0 \ \forall x \in D$. Here the first of Eqs.(3.3) always has a solution $V = \varphi(x)$; the function $\lambda_1(x)$ can be chosen fairly freely, and we take $\lambda_1(x) = 0$. Then the vector $\nabla\varphi$ is orthogonal to the vector $f_1(x)$ and its projection along the vector $f_2(x)$ does not vanish, because that would mean the collinearity of the vectors $f_1(x)$ and $f_2(x)$, which would contradict the assumption of their linear independence in the domain D. Hence we have the representation $\nabla\varphi(x) = \beta_1(x) f_1(x) + \beta_2(x) f_2(x)$, where $\beta_2(x)$ preserves its sign. When substituting $\nabla\varphi(x)$ into the second Eq.(3.3) we obtain the relation

$$\beta_1(x)(f_1(x), f_2(x)) + \beta_2(x) f_2^2(x) = \lambda_2(x) \varphi(x) + G(x)$$

which is satisfied when $\lambda_2(x) = 0$, $G(x) = \beta_2(x)[f_2(x) - f_1^{-2}(x)(f_1(x), f_2(x)) f_1(x)]^2$.

Because the function $G(x)$ preserves its sign, the conditions of Theorem 5 are satisfied, so that with the assumptions that have been made an OM exists and the system is non-controllable.

Example. Consider the system

$$x^{\cdot} = -y + x(u - u_0)^2, \quad y^{\cdot} = x + y(u - u_0)^2 \tag{3.4}$$

We have

$$f(x, u) = \begin{pmatrix} -y \\ x \end{pmatrix} + (u - u_0)^2 \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. $\alpha_1 = 1 > 0$, $\alpha_2 = (u - u_0)^2 \geqslant 0$ and $\det(f_1, f_2) = -x^2 - y^2$.

Thus system (3.4) is non-controllable in any domain not containing the origin of coordinates.

Consideration of system (3.3) for this case shows that system (3.4) is non-controllable in any domain, because system (3.3) has a solution

$$V = x^2 + y^2 - c \ (\lambda_1 = \lambda_2 = 0, \ G = 2(x^2 + y^2))$$

which, by an appropriate selection of the constant c, defines an OM N in any domain D.

4. The controllability of linear systems with restrictions on the control. The presence of restrictions on the control changes the property of controllability even for linear autonomous systems, for which the Kalman criterion already becomes insufficient, with the controllability of the system depending on the nature of the restriction, whether it is geometrical, integral or mixed.

Using Eqs.(2.1) we study the controllability of the autonomous system

$$x^{\cdot} = Ax + bu \quad (x \in R^n, u \in R^1, n \geqslant 2) \tag{4.1}$$

with the geometrical restrictions

$$|u| < u_0 \tag{4.2}$$

We will assume that system (4.1) is written in a basis in which the matrix A has real Jordan form. We consider its controllability on the assumption that without the restriction (4.2) it is controllable, i.e. satisfies the condition

$$\det(b, Ab, \ldots, A^{n-1}b) \neq 0 \tag{4.3}$$

We shall use Theorem 3. Eq.(2.1) for system (4.1) has the form

$$(Ax + bu, \nabla V) = \lambda V + G \tag{4.4}$$

We will consider three cases according to the eigenvalues of the matrix A.

We first suppose that among the eigenvalues there is a number $\lambda_1 = \alpha + i\beta$ with non-zero real part. Then the matrix A and vectors x, b can be represented in the form $A^T = (A_1^T, A_2^T)$, $x^T = (x_1^T, x_2^T)$, $b^T = (b_1^T, b_2^T)$, where $x_1, b_1 \in R^2$ and

$$A_1 = \left\| \begin{matrix} \alpha & -\beta & 0 & \ldots & 0 \\ \beta & \alpha & 0 & \ldots & 0 \end{matrix} \right\|$$

Eqs.(4.4) are satisfied if we take

$$V = -c^2 + x_1^2, \quad \lambda = 0, \; G = 2\alpha x_1^2 + 2u(b_1, x_1)$$

Using restriction (4.2) and the condition $\alpha \neq 0$ we conclude that the function G is sign-constant in the domain $D \times U$, $D = \{x: x_1^2 > \nu^2\}$, $U = \{u: |u| < u_0\}$ for $\nu^2 \geqslant u_0^2 b_1^2/\alpha^2$. According to Theorem 3 system (4.1) under restrictions (4.2) is non-controllable in the domain D, which means, also in R^m. The OM will be the domain bounded by the elliptic cylinder $x_1^2 = c$ with appropriate c.

Suppose next that amongst the eigenvalues there is a real number $\lambda_1 \neq 0$. As in the preceding case, we select a matrix $A_1 = (\lambda_1, 0, \ldots, 0)$ and vectors $x_1, b_1 \in R^1$. Eq.(4.4) is satisfied if we put $V = -c + x_1$, $\lambda = 0$, $G = \lambda_1 x_1 + ub_1$. The function G is sign-constant in the domain $D \times U$, $D = \{x: |x_1| > \nu\}$, $U = \{u: |u| < u_0\}$ for $\nu \geqslant u_0/|\lambda_1 b_1|$. By Theorem 3 system (4.1) with restrictions (4.2) is non-controllable in the domain D, which means that in R^n the OMs will be domains bounded by planes $x_1 = c$ by the appropriate choices of c.

It remains to consider the case when all eigenvalues are purely imaginary or zero. Here the functions G found in the preceding cases are no longer sign-constant and the investigation of Eq.(4.4) is much more difficult. Furthermore, it turns out that system (4.1) in this case is controllable in R^n, which means that a solution of Eq.(4.4) with the properties specified by Theorem 3 does not exist.

To prove the controllability we shall show the existence of a control solving the specific two-point problem $x(t_0) = x_0$, $x(t_1) = x_1$ for system (4.1). We require that the trajectories lie in a sphere of radius $R = 2\max(\|x_0\|, \|x_1\|)$. Then using the satisfaction of conditions (4.3) one can introduce a new control $v = c^T x + u$, such that for $\|x\| < R$ the condition $\|v\| \leqslant v_0$ is satisfied, where $0 < v_0 < u_0$, and all the eigenvalues of the matrix $A - bc^T$ are purely imaginary, $\lambda_j = i\omega_j$, (the ω_j being incommensurable, $j = 1, \ldots, k$) for even $n = 2k$, while for odd $n = 2k + 1$ we have $\lambda_j = i\omega_j$ (ω_j incommensurable, $j = 1, \ldots, k$), and $\lambda_{2k+1} = 0$.

Instead of system (4.1) with restrictions (4.2) we will consider the system $x^{\cdot} = A_c x + bv$ ($A_c = A - bc^T$) with restrictions $|v| \leqslant v_0$, assuming that the matrix A_c has real Jordan form.

We will first consider the case of even $n = 2k$. Because for $v \equiv 0$ the solution lies on a sphere, the required control is constructed in two stages. In the first stage we construct a control which takes the system from the initial sphere $x^2 = x_0^2$ to the final sphere $x^2 = x_1^2$ at some time t_*. This is always possible: because $V = x^2$ we have $V^{\cdot} = 2x^T bv$, so that $V^{\cdot} \geqslant 0$ when $v = v_0 \operatorname{sign}(x^T b)$ and $V^{\cdot} \leqslant 0$ when $v = -v_0 \operatorname{sign}(x^T b)$. A control chosen in this way ensures that the trajectory is contained in the sphere $\|x\| \leqslant \max(\|x_0\|, \|x_1\|)$. After this, with zero control we arrive at some finite time t_1 at the final point x_1 if all the quantities $\Delta\varphi_j/\omega_j$ are commensurate, or in a neighbourhood of x_1 that can be chosen to be as small as desired if the $\Delta\varphi_j/\omega_j$ are incommensurable. Here $\Delta\varphi_j = \varphi_j(t_1) - \varphi_j(t_*)$ are the increments in polar angles of the points $x_1, x(t_*)$ in the planes corresponding to eigenvalues λ_j. In the latter case, by a theorem from /7/ $\forall\varepsilon > 0$ there exists a control v satisfying the restriction $|v| < \varepsilon$ taking the system from the initial point $x(t_*)$ to any point of some neighbourhood of the point

$x(t_*)\exp A_c(t_1-t_*)$, which includes the point x_1. This proves the existence of control in the given case.

For odd n, unlike in the preceding case, in the first stage it is necessary to arrive at time t_* not just at the final sphere $x^2=x_1^2$, but also to ensure that the condition $x_{2k+1}(t_*)=x_{2k+1,1}$ is satisfied. To do this one must combine the control found in the preceding case with zero control, choosing the switch-over times in a corresponding manner. In the final stage the proof is as in the preceding case.

We have thus proved the following theorem.

Theorem 6. System (4.1) is controllable in R^n with restriction (4.2) only in the case when condition (4.3) is satisfied and all eigenvalues of the matrix A are purely imaginary (which includes zero eigenvalues), and the quantity u_0 can be taken to be as small as required.

5. ***Estimates of the controllable domain.*** It was shown above that a controllable linear system can become non-controllable (in the entire space) when geometric restrictions are imposed on the control. OMs constructed for cases of non-controllability are separated from zero, and the system will be controllable in some domain containing the origin of coordinates because it is known /7/ that the domain of controllability contains some neighbourhood of the origin of coordinates. From this there arises the problem of describing the domain of controllability, the solution of which is however very difficult even for two-dimensional systems /5/. The analysis of oriented manifolds developed above allows one to find a radius R_0 such that for $R>R_0$ system (4.1) is non-controllable on the sphere $\|x\|<R$ for $|u|\leqslant u_0$. This characteristic is useful for describing domains of controllability and can be used in a quantitative estimate of the quality of the control system.

We will compute R_0 for system (4.1) with restriction (4.2). Suppose the matrix A has k real eigenvalues λ_j, (including k^+ positive eigenvalues λ_j^+ and k^- negative eigenvalues λ_j^-) and m complex eigenvalues $\varkappa_j=\alpha_j+i\beta_j$ (including m^+ with positive real parts α_j^+ and m^- with negative real parts α_j^-). The coordinates corresponding to the first rows of the associated real Jordan cells will be denoted by $x_j^\pm$ and $\xi_j^\pm$, $\eta_j^\pm$, and the corresponding coordinates of the vector b by $p_j^\pm$, and $\gamma_j^\pm$, $\delta_j^\pm$. The boundaries of the oriented manifolds are obtained by setting the following functions to zero:

$$V_{1\nu}^\pm=-c^2+\sum_{j=1}^{\nu}x_j^{\pm 2},\quad V_{2\mu}^\pm=-c^2+\sum_{j=1}^{\mu}(\xi_j^{\pm 2}+\eta_j^{\pm 2})$$
$$V_{\nu j}^\pm=c^2+x_\nu^{\pm 2}-x_j^\mp\quad(\nu,j=1,\ldots,k^\pm,\ \mu=1,\ldots,m^\pm)$$

with the condition that the functions

$$G_{1\nu}^\pm=\dot V_{1\nu}^\pm=2\sum_{j=1}^{\nu}(\lambda_j^\pm x_j^{\pm 2}+p_j^\pm x_j^\pm u)$$
$$G_{2\mu}^\pm=\dot V_{2\mu}=2\sum_{j=1}^{\mu}[\alpha_j^\pm(\xi_j^{\pm 2}+\eta_j^{\pm 2})+u(\xi_j^\pm\gamma_j^\pm+\eta_j^\pm\delta_j^\pm)]$$
$$G_{\nu j}^\pm=\dot V_{\nu j}=2\lambda_\nu^\pm x_\nu^{\pm 2}-\lambda_j^\mp x_j^\mp+u(2x_\nu^\pm p_\nu^\pm-p_j^\mp)$$

are sign-constant. (In the quoted formulae one uses only the upper signs or the lower signs). From the sign-constancy condition on the functions $G_{1\nu}^\pm$ and $G_{2\mu}^\pm$ over all space, and the functions $G_{\nu j}^\pm$ on the manifolds $V_{\nu j}^\pm=0$ we obtain

$$R_0=\min(R_{1\nu}^\pm,R_{2\mu}^\pm,R_{\nu j}^\pm)\tag{5.1}$$
$$R_{1\nu}^\pm=u_0\max_{x^\pm\in S_\nu}\Big|\sum_{j=1}^{\nu}p_j^\pm x_j^\pm\Big/\sum_{j=1}^{\nu}\lambda_j^\pm x_j^{\pm 2}\Big|$$
$$R_{2\mu}^\pm=u_0\max_{(\xi^\pm,\eta^\pm)\in S_{2\mu}}\Big|\sum_{j=1}^{\mu}(\xi_j^\pm\gamma_j^\pm+\eta_j^\pm\delta_j^\pm)\Big/\sum_{j=1}^{\mu}\alpha_j^\pm(\xi_j^{\pm 2}+\eta_j^{\pm 2})\Big|$$
$$R_{\nu j}^\pm=u_0[(p_\nu^{\pm 2}u_0+|p_j^\mp(2\lambda_\nu^\pm-\lambda_j^\mp)|)/|\lambda_j^\mp(2\lambda_\nu^\pm-\lambda_j^\mp)|]$$

(where S_k is the unit sphere with centre at the origin of coordinates).

6. ***Control of the angular motion of a rigid body.*** It is well-known that the angular motion of a rigid body moving inertially can be controlled by an "obliquely-positioned" reactive thruster /1, 8/. We will consider the properties of these controllable motions when

there is a restriction on the control. We will linearize the equations of motion in a neighbourhood of uniform motion about the third principal axis with zero control:

$$\omega_1^{\cdot} = a_1\omega\omega_2 + \alpha_1 u, \quad \omega_2^{\cdot} = a_2\omega\omega_1 + \alpha_2 u, \quad \omega_3^{\cdot} = \alpha_3 u$$
$$a_1 = (A_2 - A_3)/A_1, \quad a_2 = (A_3 - A_1)/A_2 \qquad (6.1)$$

Here ω_1, ω_2 and ω_3 are perturbations of the projections of the angular velocity vector onto the principal axes, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a vector characterizing the direction of the torque of the reactive force, u is the magnitude of the reactive force, ω is the angular velocity of the uniform rotation, and A_1, A_2, A_3 are the principal moments of inertia of the body.

We will investigate the controllability of system (6.1) with the restriction $|u| \leqslant u_0$ on the control and the assumption that when there are no restrictions system (6.1) is controllable, i.e.

$$\det(b, Ab, A^2b) = a_1a_2\alpha_3\omega^3(a_1\alpha_2^2 - a_2\alpha_1^2) \neq 0$$

Depending on the roots λ_1, λ_2 and λ_3 of the characteristic equation

$$\det(A - \lambda E) = -\lambda(\lambda^2 - a_1a_2\omega^2) = 0$$

we consider two cases:

1) rotation around the smaller and larger axis of the inertial ellipsoid

$$(a_1a_2 < 0){:}\ \lambda_1 = i\omega\sqrt{-a_1a_2}, \quad \lambda_2 = -i\omega\sqrt{-a_1a_2}, \quad \lambda_3 = 0$$

2) rotation around the middle axis of the inertial ellipsoid $(a_1a_2 > 0)$:

$$\lambda_1 = \omega\sqrt{a_1a_2}, \quad \lambda_2 = -\omega\sqrt{a_1a_2}, \quad \lambda_3 = 0$$

According to Theorem 6 system (6.1) is controllable in the first case in all space for any value of u_0.

In the second case system (6.1) is non-controllable for $|u| \leqslant u_0$ in all space, and for estimates of the domain of controllability we use formulae (5.1), first putting system (6.1) into Jordan form.

From formulae (5.1) we find

$$R_0 = \min(R_1^+, R_1^-, R_{11}^+, R_{11}^-)$$
$$R_1^{\pm} = u_0 \,|\alpha_1\sqrt{a_2} \pm \alpha_2\sqrt{a_1}\,|/(2a_1a_2\omega), \quad R_{11}^{\pm} = R_1^{\pm} + R_1^{\mp}/3$$

Finally we have

$$R_0 = \min(R_1^+, R_1^-) \qquad (6.2)$$

Formulae (6.2) can be used, for example, to estimate the quality of a control system for a rigid body and to optimize its parameters $(\alpha_1, \alpha_2, \alpha_3)$ if one takes $\varkappa = \max R_0$ to be the quality criterion. For a fuller estimate it is also necessary to consider forms of motion other than uniform rotation.

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Translated by R.L.Z.